Admissions Testing Service

STEP Solutions 2016

Mathematics

STEP 9465/9470/9475

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Test

STEP III 2016 Solutions

1. Part (i) is most simply dealt with by the suggested method, change of variable, and it is worth completing the square in the denominator to simplify the algebra leading to a trivial integral. Part (ii) can either be attempted immediately using integration by parts, starting from I_n and obtaining $\int_{-\infty}^{\infty} \frac{2nx(x+a)}{(x^2+2ax+b)^{n+1}}\, dx$ and then writing the numerator as $2n(x^2+2ax+b) - na(2x+2a) - 2n(b-a^2)$. Alternatively, use of the same substitution in I_{n+1} as in part (i) leads to the need to integrate $\cos^{2n} u$, which in turn can be written as $\cos^{2n-2} u(1-\sin^2 u) = \cos^{2n-2} u - \cos^{2n-2} u \sin u. \sin u$, with the second term being susceptible to integration by parts. Part (iii) follows from the previous parts by induction using part (ii) to achieve the inductive step and (i) the base case.

2. There are numerous correct ways through this question. Working parametrically with $x = at^2$, $y = 2at$ gives a normal as $tx + y = at^3 + 2at$ and imposing that this passes through $(ap^2, 2ap)$ yields $t^2 + tp + 2 = 0$ (*) which has roots q and r, the former giving (i). As a consequence, $r + q = -p$ and $rq = 2$, so that QR, $2x - (r+q)y + 2aqr = 0$, simplifies to $2x + py + 4a = 0$, and thus passes through $(-2a, 0)$ for (ii). T can be shown to be $\left(-a, \frac{-2a}{p}\right)$, which, of course, lies on $x = -a$, and as (*) had two real distinct roots, q and r, $p^2 - 8 > 0$, which yields $\left|\frac{-2a}{p}\right| < \frac{a}{\sqrt{2}}$.

3. Differentiating, multiplying by denominators and dividing by the exponential function, gives $[Q(P+P') - PQ'](x+1)^2 = (x^3-2)Q^2$ which, invoking the factor theorem, gives the first required result. Denoting the degree of P by p and that of Q by q in this expression yields $p + q + 2 = 2q + 3$ and hence the desired result in (i). Furthermore, in the given case, substitution in the same result and postulating $P(x) = ax^2 + bx + c$ yields consistent equations for a, b and c and thus $P(x) = x^2 - 2x$.
For part (ii), commencing as in part (i) demonstrates again that Q has a factor $(x+1)$ as $[Q(P+P') - PQ'](x+1) = Q^2$. Supposing $Q(x) = (x+1)^n S(x)$, where $n \geq 2$ and $S(-1) \neq 0$, with $P(-1) \neq 0$ and substituting in the expression already derived leads to a contradiction.

4. The considered expression equates to $\frac{(x-1)x^r}{(1+x^r)(1+x^{r+1})}$ and so, by the method of differences,

$\sum_{r=1}^{N} \frac{x^r}{(1+x^r)(1+x^{r+1})} = \frac{1}{(x-1)}\left[\frac{1}{1+x} - \frac{1}{1+x^{N+1}}\right]$, and letting $N \to \infty$, the desired result is obtained. Writing $\operatorname{sech}(ry)$ as $\frac{2e^{-ry}}{1+e^{-2ry}}$ and similarly $\operatorname{sech}\big((r+1)y\big)$, the result of part (i) with $x = e^{-2y}$ can be used to obtain the result. Care needs to be taken to write $\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry) \operatorname{sech}\big((r+1)y\big)$ as

$2\left[\sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}\big((r+1)y\big) + \operatorname{sech} y\right]$ which with the previous deduction of (ii) can be simplified to $2 \operatorname{cosech} y$.

5. The binomial expansion, evaluated for $x = 1$, appreciating that terms are symmetrical contains two terms equal to the LHS of the inequality, and so truncating to them gives double the required result in (i). Appreciating that $\frac{(2m+1)!}{(m+1)!m!}$ is an integer and that if $m + 1 < p \leq 2m + 1$, with p a prime, implies p divides the numerator and not the denominator of this expression and hence divides the integer then can be extended for all such primes yielding the result, with the deduction following from (i). For (iii), it can be shown that $m + 1 \leq 2m$ and writing $P_{1,2m+1}$ as

$P_{1,m+1}P_{m+1,2m+1}$, combining the given result and (ii), the desired result is obtained. Part (iv) is obtained by use of strong induction with the supposition, $P_{1,m} < 4^m$ for all $m \leq k$ for some particular $k \geq 2$, and considering the cases k even and odd separately and making use of (iii).

6. Using $R\cosh(x+\gamma) = R(\cosh x \cosh\gamma + \sinh x \sinh\gamma)$, $R = \sqrt{B^2 - A^2}$ and $\gamma = \tanh^{-1}\frac{A}{B}$ if $B > A > 0$. If $B = A$, then $A\sinh x + B\cosh x = Ae^x$. If $-A < B < A$, the expression can be written as $R\sinh(x+\gamma)$ with $R = \sqrt{A^2 - B^2}$ and $\gamma = \tanh^{-1}\frac{B}{A}$. If $B = -A$, then $A\sinh x + B\cosh x = -Ae^{-x}$, and if $B < -A$, the expression can be written as $R\cosh(x+\gamma)$ with $R = -\sqrt{B^2 - A^2}$ and $\gamma = \tanh^{-1}\frac{A}{B}$. For part (i), solving simultaneously gives $a\sinh x + b\cosh x = 1$, which gives the desired solutions using the first result of the question. Similarly for part (ii) using the appropriate result, $x = \sinh^{-1}\left(\frac{1}{\sqrt{a^2-b^2}}\right) - \tanh^{-1}\frac{b}{a}$. For (iii), we require that the conditions for (i) give two solutions, i.e. that $b > a$ and $\left(\frac{1}{\sqrt{b^2-a^2}}\right) > 1$, and so $a < b < \sqrt{a^2+1}$, and vice versa, if this applies there are indeed two solutions. For (iv), we require case (i) to give coincident solutions, i.e. $b = \sqrt{a^2+1}$ and hence $x = -\tanh^{-1}\frac{a}{\sqrt{a^2+1}}$, and so $y = \frac{1}{\sqrt{a^2+1}}$. The reverse argument also applies.

7. Considering $(\omega^r)^n$ establishes by the factor theorem that each factor on the LHS is a factor of the RHS, and comparing coefficients of z^n between the two sides establishes that no numerical factor is required. For part (i), representing X_r by ω^r , then there are two cases to consider, P will be represented either by $re^{\frac{\pi i}{n}}$, or $re^{\left(\frac{\pi}{n}+\pi\right)i}$. The product of moduli is the moduli of the product of factors, and the product of the factors can be simplified using the stem and choosing z in turn as the representations of P to give the required result in both cases. Proceeding similarly for *n* odd, the first case yields $|OP|^n + 1$, and the second, $|OP|^n - 1$, if $|OP| \geq 1$, and $1 - |OP|^n$ if $|OP| < 1$. Using the same representations for the X_r in part (ii), and the same technique with the moduli, the stem can be divided by $(z-1)$ to give $(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1}) = z^{n-1} + z^{n-2} + \cdots + 1$ which then gives the desired result when $z = 1$.

8. The first result in (i) is obtained by the substitution $x = -u$ (followed by a second $u = x$!). Substituting for $f(-x)$ in the initial statement using the result obtained readily leads to $f(x) = x$ which is simply verified. Alternatively, subtracting the result from the initial equation leads to $f(x) = f(-x)$ which substituting gives the required result again. In part (ii), substituting $K(x)$ for x in the equation for $g(x)$ gives an equation for $g\left(\frac{x+1}{x-1}\right)$ which can be substituted in the equation to be solved to give the desired result. Similarly, in part (iii), substituting $\frac{1}{1-x}$ for x gives an equation for $h\left(\frac{1}{1-x}\right)$ and $h\left(\frac{x-1}{x}\right)$, and then repeating this substitution in the equation just obtained gives an equation for $h\left(\frac{x-1}{x}\right)$ and $h(x)$. Adding the given and last equations and subtracting that first found leads to $h(x) = \frac{1}{2} - x$.

9. There are numerous ways to obtain $PX = \frac{2}{\sqrt{3}}a$ via e.g. knowledge of the centroid of a triangle, Pythagoras' theorem, trigonometry or a combination of these, leading to the initial extension result. Pythagoras' theorem can be used to find RX and hence the given tension. The equation of motion in the direction XP combines the tension in PX and the resolved parts of the tensions in the other two springs. Writing the cosine of the angle between RX and PX produced as

$\frac{\frac{1}{\sqrt{3}}a+x}{\sqrt{a^2+\left(\frac{1}{\sqrt{3}}a+x\right)^2}}$, leads to $-\lambda-\frac{3\lambda}{l}x+2\lambda\left(\frac{1}{\sqrt{3}}a+x\right)\frac{\sqrt{3}}{2a}\left(1+\frac{\sqrt{3}}{2}\frac{x}{a}+\frac{3x^2}{4a^2}\right)^{\frac{-1}{2}}=m\ddot{x}$ which making an approximation for small x and the binomial expansion leads to $-\frac{3\lambda}{4la}\left(4a-\sqrt{3}l\right)x=m\ddot{x}$, and hence the final result.

10. Resolving along a line of greatest slope initially, bearing in mind the acceleration due to circular motion, gives an expression for the initial tension in the string which can be substituted in the expression obtained for normal contact force obtained by resolving perpendicular to the slope. Requiring a positive normal contact force then gives the desired result. To complete circles, there must be a tension in the string when the particle is at the highest point it can reach on the plane. Conserving energy gives $v^2=u^2-4ag\cos\beta\sin\alpha$ and resolving down the plane yields $T\cos\beta+mg\sin\alpha=m\frac{v^2}{a\cos\beta}$ resulting in $u^2>5ag\cos\beta\sin\alpha$; this combined with the first result will give the final desired result. (The first result can be found elegantly by resolving perpendicularly to the string.)

11. In part (i), expressing the resistance as kv , then the zero acceleration condition gives $k=\frac{P}{16U^2}$. Writing the equation of motion using $a=v\frac{dv}{dx}$, and solving the differential equation by separating variables, the integration gives $X_1=\left[\frac{m}{k}\left(2U\ln\left(\frac{4U+v}{4U-v}\right)-v\right)\right]_U^{2U}$ which evaluated and rearranged is the required result. Part (ii) follows a similar route, instead expressing the resistance as kv^2 , with $k=\frac{P}{64U^3}$. The same technique for the differential equation gives a slightly simpler integration to yield the result. $\lambda X_1-\lambda X_2$ can be manipulated to be $\frac{4}{3}\ln 24-2\ln 5-1$ which can be shown to be positive using the appropriate bounds and so answering part (iii) that X_1 is the larger.

12. Using the binomial distribution, $\mu=20n$, $\sigma^2=16n$, writing $16n\leq X\leq 24n$ as $|X-20n|\leq 4n$ enables Chebyshev to be applied with $k=\sqrt{n}$ leading to the required result in (i). Similarly, in part (ii), considering a Poisson distribution with mean n , and appreciating that $|X-n|>n$ implies $X>2n$ in these circumstances, the same value of k as in part (i) with Chebyshev leads to the desired result.

13. Showing that $X-a$ has the same kurtosis as X requires the expectations of $X-a$, $(X-a-\mu+a)^2$, and $(X-a-\mu+a)^4$ to be obtained and substituted. For part (i), the numerator can be obtained by an integration by parts reducing the integral to the one that gives the variance. Expanding T^4 as $\sum\left({Y_r}^4+4{Y_r}^3Y_s+6{Y_r}^2{Y_s}^2+12Y_sY_t{Y_r}^2+24Y_rY_sY_tY_u\right)$, where the summation is over all values without repetition, and taking the expectation of these terms gives the requested result in part (ii). Defining $Y_i=X_i-\mu$, the kurtosis of Y_i by the first result gives $E\left({Y_i}^4\right)=(3+\kappa)\sigma^4$ and defining T as in (ii), the kurtosis of $\sum_{i=1}^{n}X_i$ is, using the result of (ii), $\frac{n(3+\kappa)\sigma^4+3n(n-1)\sigma^2\sigma^2}{n^2\sigma^4}-3$ giving the required answer.